Khovanov homology for virtual knots with arbitrary coefficients

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Abstract

We construct explicitly the Khovanov homology theory for virtual links with arbitrary coefficients by using the twisted coefficients method. This method also works for constructing Khovanov homology for "non-oriented virtual knots" in the sense of [Viro], in particular, for knots in $\mathbf{R}P^3$.

Virtual knots were introduced in mid-nineties by Lou Kauffman, see [KaV]. By a virtual diagram we mean a four-valent graph on the plane endowed with a special structure: each crossing is either said to be classical (in this case one indicates which pair of opposite edges at this crossing forms an overcrossing; the remaining two edges form an undercrossing) or virtual (in this case, we do not specify any addtional structure; virtual crossings are just marked by a circle). Two virtual diagrams are called *equivalent* if one of them can be obtained from the other by a finite sequence of generalized Reidemeister moves and planar isotopies. Recall that the list of generalized Reidemeister moves consists of classical Reidemeister moves (see,e.g., [Man]) and the detour move. The latter means that if wa have a purely virtual arc containing only virtual crossings, we may remove it and restore in any other place of the plane, see Fig. 1.

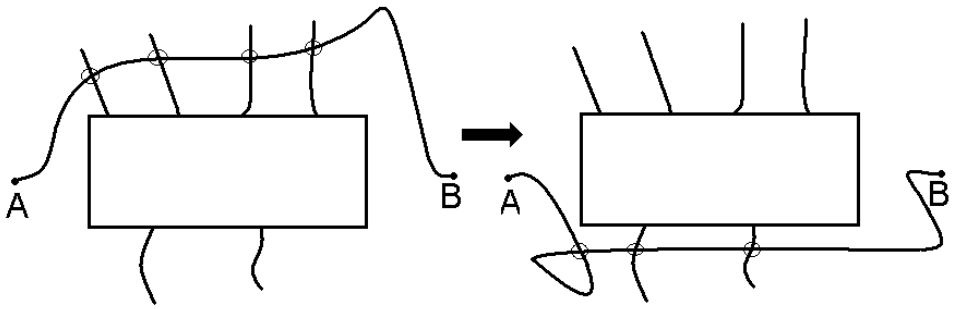

Figure 1: The detour move

One of the most important achievements in modern knot theory is the Khovanov categorification of the Jones polynomial proposed in [Kh]. The main idea is to replace (Laurent) polynomials by homology groups of bigraded vector spaces. Thus, with each link diagram, one associates a bigraded algebraic complex; all cohomology groups of this complex are knot invariants, whence its graded Euler characteristic coincides with the Jones polynomial (in a certain normalization).

Later on, we deal with bigraded complexes of the type $\mathcal{C} = \oplus_{i,j}\mathcal{C}^{i,j}$ with *height* i and *grading* j. The differential ∂ in such a complex preserves the grading and increases the height by one.

For a graded linear space A define its *graded dimension qdim* by $qdimA = \sum_i q^i \cdot dimA^i$, where A^i is the subspace of A generated by all vectors of grading i. Chain spaces $\mathcal{C}^j$ of $\mathcal{C}$ of a given height j can be treated as such graded spaces.

It would be more conveninent to speak about *cohomology*, but we should rather say *Khovanov homology* instead and say *chains, cycles, boundaries* rather than *cochains, cocycles, coboundaries*. For such complexes there are well defined degree shift and height shift operations: $\mathcal{C} \mapsto \mathcal{C}[k,l]$; where $(\mathcal{C}[k,l])^{i,j} = \mathcal{C}^{i-k,j-l}$. By the *graded Euler chararcteristic* of a complex $\mathcal{C}^{ij}$ we mean the alternating sum of graded dimensions of the chain spaces, or, equivalently, that of homology groups. For the chain spaces we have: $\sum_i (-1)^i qdim\mathcal{C}^i = \sum_{i,j}(-1)^i q^j dim\mathcal{C}^{i,j}$.

A beautiful property of the Khovanov homology is its *functoriality* meaning the following. Each cobordism in $\mathbf{R}^3 \times I$ between two links $K_1 \subset \mathbf{R}^3 \times \{0\}$ and $K_2 \subset \mathbf{R}^3 \times \{1\}$, generates a natural mapping between the Khovanov homology $Kh(K_1) \to Kh(K_2)$ which is invariant under isotopy. Here the Khovanov homology is built of bricks, which are circles on the plane, and the complex cobordism category can be viewed as an algebraic counterpart of these circles where maps are represented by cobordisms.

The functoriality of the Khovanov homology was proved by Jacobsson in the algebraic setup and then by Bar-Natan in the topological setup, see [BN2, Jac].

Another definition of the Khovanov homology is *local*: instead of considering states and counting the number of circles in each of them, one can construct a complex by using Matrix factorizations at every vertex, as in [KhR1], and then contracting them into the whole complex. The homology proposed in [KhR1] and [KhR2] give a categorification of the HOMFLY polynomial, the Jones 1-variable polynomial being a partial case of it.

This way is, in some sense, easier to check the invariance (because the proof becomes local), but it is much more complicated for explicit calculations. However, it is said on page 32 of [KhR1] "The network does not even have to be planar, and does not need to be embedded anywhere. In our paper, however, all such diagrams are going to be planar". This means that the construction of [KhR1] [not only for the usual $sl(2)$-homology, but in the general case] generalizes for the case of virtual knots. Indeed, all their proofs are local, and the construction of embedded graph does not feel nugatory handles of a 2-surface, thus being invariant under (de)stabilizations.

However, this construction is too much implicit. In this paper, we present an explicit construction of this homology together with several generalizations

of it.

It turns out, for example, that the Khovanov homology for virtual knots constructed in this way is invariant under virtualizations thus leading to a homology theory of "twisted virtual knots".

One can also establish the analogues of some other theories: Lee's theory, Wehrli's spanning tree expansion, etc.

In the previous papers [Ma1, Ma2], the author constructed the Khovanov theory for virtual knots. More precisely, the Khovanov complex was well defined for all virtual link diagrams only over $\mathbf{Z}_2$; the Khovanov homology with arbitrary coefficients was defined only in the case of virtual knots corresponding to so-called *orientable atoms*: **atom plays the key role in defining the Khovanov homology for virtual links, with its non-orientability being the main obstruction**. For non-orientable atoms, we presented two geometrical construction transforming them to orientable atoms, and then proving that the Khovanov homology of the target knot (atom) is an invariant of the source one.

In the present note, we will, for arbitrary coefficient ring, construct a differential with $\partial^2 = 0$ explicitly. The key ideas are to use the **twisted coefficient** methods: one should change the basis of the Hopf algebra (representing the Khovanov homology of the unknot) while passing from one classical crossing to another crossings and to use the exterior product of tensor spaces instead of the usual symmetric product.

Thus, with each virtual link diagram we associate a bigraded complex with homology being invariant under generalized Reidemeister moves. We wish to point out the main properties of this construction.

1. The complex is constructed by using *atoms*; this it is invariant under *virtualization*, the natural transformation of virtual links preserving the Jones polynomial.

2. Since there exists a map from twisted virtual knots to usual virtual knots modulo virtualization, this approach works for twisted virtual knots as well.

3. In the $\mathbf{Z}_2$-case the complex coincides with the $\mathbf{Z}_2$-complex first proposed in [Ma1].

4. For orientable atoms this complex has the same homology as the complex proposed in [Ma2].

5. The invariance proof is local and repeats that for the classical case, see, e.g., [BN], the main difficulty being the definition of the complex.

6. A partial case of this theory is theory of knots in $\mathbf{R}P^3$; recall that the Kauffman bracket version of the Jones polynomial for such knots was proposed in [Dro].

1 The Kauffman bracket and the Jones polynomial. Atoms. Twisted virtual knots

Consider an oriented virtual diagram L and a diagram $|L|$ obtained from L by forgetting the oritentation. Let us smooth classical crossings of $|L|$ according to the following rule. Each classical crossing can be *smoothed* in one of two ways, the positive way A and the negative way B.

A *state* is a choice of smoothing for all classical crossings of a diagram. Each state generates a set of curves on the plane having only virtual crossings. In other words, we get some virtual unlink. Suppose the diagram L has n classical crossings. Enumerate them arbitrarily. We get 2^n states which can be encoded by vertices of the n-dimensional discrete cube $\{0,1\}^n$, where 0 and 1 correspond to A-smoothing and B-smoothing, respectively. Call this cube the *state cube* of the diagram. Two states are *adjacent* if they differ in a unique coordinate. Any two adjacent vertices are connected by an edge of the cube. Orient all edges of the cube as the coordinate increases, i.e., from A-type smoothing to B-type smoothing. By *height* of the vertex we mean the number of B-type smoothing of the corresponding state.

For each state s, denote by $\alpha(s)$ the number of A-smoothings, let $\beta(s) = n - \alpha(s)$ be the number of B-smoothings and let $\gamma(s)$ be the number of componentsof the resulting unlink. The Jones-Kauffman polynomial is then defined as:

$$X(L) = (-a)^{-3w(L)} \sum_s a^{\alpha(s)-\beta(s)} (-a^2 - a^{-2})^{\gamma(s)-1}, \tag{1}$$

where $w(L)$ is the writhe of the oriented diagram L (i. e., the difference between the number of positive crossings and the number of negative crossings).

The unnormalized version $\sum_s a^{\alpha(s)-\beta(s)}(-a^2-a^{-2})^{\gamma(s)-1}$ is called the *Kauffmab bracket*; both the Kauffman bracket and the Jones-Kauffman polynomial are (Laurent) polynomials in one variable. The Jones-Kauffman polynomial is invariant under (generalized) Reidemeister moves.

After the variable change $a = \sqrt{(-q^{-1})}$, we get instead of the Jones polynomial V its modified versions J and $\hat{J}$. They differ by a normalization on the unknot: we have $J = 1$ on the unknot whence $\hat{J} = 1$ for the empty link with no components. Herewith $J = \frac{\hat{J}}{(q+q^{-1})}$. In more details, $\hat{J}$ goes as follows. Let L be an oriented virtual diagram and let $|L|$ be the corresponding unoriented virtual diagram; denote by n_+ and n_- the number of positive (resp., negative) classical crossings of L; let n be the total number of classical crossings: $n = n_+ + n_-$. Set

$$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle, \tag{2}$$

where $\langle L \rangle$ is the modified Kauffman bracket defined axiomatically as $\langle \bigcirc \rangle = (q + q^{-1})$, $\langle L \rangle = \langle L_A \rangle - q\langle L_B \rangle$, $\langle L \sqcup \bigcirc \rangle = (q + q^{-1}) \cdot \langle L \rangle$.

Later on, we deal with the polynomial $\hat{J}$ calling it *the Jones polynomial*.

The polynomial $\hat{J}$ has an easy description in terms of the state cube. Taking away the normalization factors $(-1)^{n_-}q^{n_+-2n_-}$ we get the (slightly modified) Kauffman bracket $\sum_s(-q)^{\beta(s)}((q+q^{-1})^{\gamma(s)})$. That is, we take the sum over all vertices of the cube of $(-q)$ to the power equal to the *height of the vertex*, all multiplied by $(q+q^{-1})$ to the number of circles corresponding to the vertex of the cube. Thus, on the level of polynomials, with each circle we associate $(q+q^{-1})$ and then we multiply these polynomials, shift them by multiplying by $\pm q^k$ and collect as summands.

Thus, the Jones polynomial can be restored from the information about the *number of circles* at the states of the cube. If we also take into account the way **how these circles bifurcate** while passing from a state to the adjacent state, we get the Khovanov complex.

It turns out that all this information needed for the Jones polynomial is contained in the atom representing its planar diagram.

An *atom* is a pair (M,Γ) : closed two-manifold M and a four-valent graph (frame) $\Gamma\subset M$ dividing M into black and white cells together with a checkerboard coloring of these cells. Atoms are considered up to natural combinatorial equivalence: diffeomorphisms of the underlying manifolds preserving the frame and the cell coloring.

Associate with a given virtual link diagram L the atom $V(L)$ to be constructed as follows. The vertices of $V(L)$ are in one-to one correspondence with *classical* crossings of L. These classical crossings are connected by *branches of the diagram*, which might intersect on the projection plane. For each classical crossing we have four emanating branches. With these edges we associate four edges of the atom frame to connect the corresponding vertices. The rule for attaching black (resp., white) cells is defined by the planar diagram L. Namely, let X be a classical crossing of L. Enumerate the four emanating edges by x_1,x_2,x_3,x_4 in the clockwise direction in such a way that the branches x_1 and x_3 form an undercrossing whence x_2 and x_4 form an overcrossing. Then for attaching the black cells we chose those pairs of half-edges of the atom corresponding to (x_1,x_2) and (x_3,x_4).

Evidently, the whole information about the number of circles in states of the diagram can be extracted from the corresponding atom. In other words, the state cube can be completely restored from the atom.

By *virtualization* we mean the local transfomation in the neighbourhood of a classical crossing shown in Fig 2. Note that the virtualization does not change the state cube; it does not change the atom, either. Moreover, two diagrams corresonding to the same atoms, can be obtained from each other by a sequence of detours and virtualizations.

We shall construct the Khovanov complex starting from the atom corresponding to a given diagram. Thus the homology will be automatically invariant under virtualization. This supports the conjecture stated in [FKM] that if two calssical crossings are equivalent by a chain of generalized Reidemeister moves, detours and virtualizations, then the corresponding links are isotopic in the usual sense.

Twisted virtual knots [Viro] are close relatives of virtual knots. They are rep-

Figure 2: Two variants of virtualization

resented by knots in oriented thickenings of not necessarily orientable surfaces modulo stabilization/destabilization.

Namely, having a non-orientable surface S, one can construct the canonical oriented I-bundle over it, which is a 3-manifold $S\tilde{\times}I$ with boundary.

A nice example of such a thickened surface is $\mathbf{R}P^2\tilde{\times}I$, which is homeomorphic to $\mathbf{R}P^3$. Thus, constructing a Khovanov homology for such knots, we shall get a Khovanov homology theory for knots in $\mathbf{R}P^3$.

Given a surface S, one can project any knot (link) from $S\tilde{\times}I$ to S. In the general position, one gets a 4-graph. In order to restore the link, one should indicate for which crossings how the two branches behave. In the orientable case, one just indicates, which branch should be over, and which branch should be under. However, in the non-orientable case this indication is relative: while walking along a non-orienting circuit, the direction upwards changes to the direction downwards.

However, such surfaces perfectly agree with atoms. Indeed, fix (once forever) an orientation of $S\tilde{\times}I$. Now, for link diagram in S, we already have a frame of the atom: a 4-valent graph with A-structure.

Now, the way for attaching black cells is the following: for a vertex X, take two emanating edges a and b. In the neighbourhood of X, denote the small vector going from the edge a to the edge b. Now, if the orientation (a, b, c) coincides with the orientation of our 3-manifold, then the angle generated by the vectors a and b is black, as well as the angle opposite to it; the remaining to angles are white.

Alternatively, if the orientations disagree, the angle between a and b is white.

Note that this choice does not depend on the ordering of a and b, nor on their directions.

This leads to the following

Theorem 1. *There is a well-defined map from twisted virtual knots to virtual knots modulo virtualization.*

Knots in such surfaces were considered by Asaeda, Przytycki and Sikora in [APS], and Viro [Viro]. In [APS] a Khovanov homology theory for such surfaces was constructed by using an additional topological information coming from surfaces.

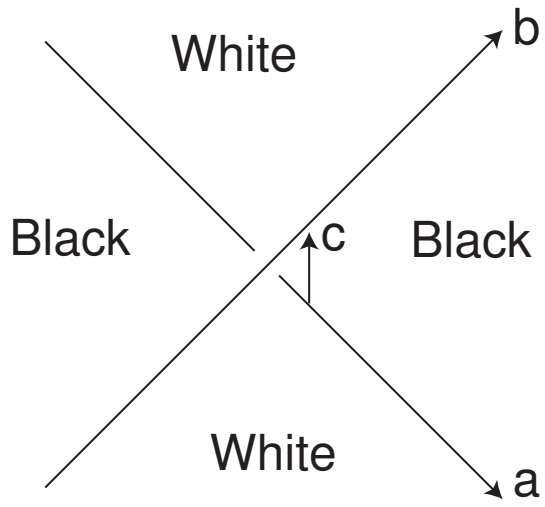

Figure 3: Constructing an atom from a diagram

From Theorem 1 it follows immediately that the Khovanov homology constructed below generalizes for twisted virtual knots.

2 Definition of the Khovanov homology

Our aim is to define a homology theory for virtual knots (orientable and unorientable) over arbitrary ring in such a way that:

1. The homology we are defining is invariant under Reidemesiter moves
2. For the case of orientable atoms (alternatible knots) this homology theory coincides with the one constructed in [Ma2].
3. The $\mathbf{Z}_2$-restriction of this homology theory coincides with the theory constructed in [Ma1].

In [Ma2] we constructed a homology theory for orientable virtual knots with arbitrary coefficients. The main obstruction to extend this theory over unorientable atoms is the possibility of $1 \to 1$ bifurcation on an edge of the cube.

If no such bifurcations occur, we may construct the Khovanov cube by using the standard differentials, the multiplication m (for $2 \to 1$-bifurcations) and the comultiplication Δ (for $1 \to 2$-bifurcations).

The situation with $1 \to 1$ bifurcation makes the problem more complicated. Indeed, if we wish to construct a grading-preserving theory without introducing any new grading, this differential should be identically equal to zero because of the dimension reasons (there should be a mapping from V to V that lowers the grading by one).

If we set this differential to be equal to zero with all other differentials (m and Δ) defined in the standard way, we get a straightforward generalization for the $\mathbf{Z}_2$ case.

The case of $\mathbf{Z}$-coefficients is more delicate: a straightforward generalization makes some 2-faces of the cube commute and some other ones anti-commute.

Notational agreement. Given several vector spaces $V_1, \dots, V_n$. We shall distinguish between two types of tensor products, the **ordered one** and the **unordered one**. For any permutation of indices $\sigma_1, \dots, \sigma_n$, we may consider the tensor product $V_{\sigma_1} \otimes \dots V_{\sigma_n}$.

In the unordered case, we identify all these tensor products by $x_{\sigma_j} \otimes \dots \otimes x_{\sigma_n} = x_1 \otimes \dots \otimes x_n$, where $x_k \in V_k$. In the second case, we set $x_{\sigma_j} \otimes \dots \otimes x_{\sigma_n} = sign(\sigma) x_1 \otimes \dots \otimes x_n$.

We shall denote such a tensor product of elements by the usual tensor product sign $x_1 \otimes x_2 \otimes \dots \otimes x_n$ in the first case and by using the exterior product sign $x_1 \wedge x_2 \wedge \dots \wedge x_n$.

Remark 1. *To avoid confusion, note that, writing $X \wedge X$, we always assume that the first X and the second X belong to different spaces; thus $X \wedge X$ is not zero; one should rather write $X_1 \wedge X_2 = -X_2 \wedge X_1$, where X_i here means the element X lying in the space V_i.*

To handle it and to make the whole cube anti-coomutative, we have to add two ingredients, sensitive to orientability of the atom:

1. With each circle of any state, we associate a vector space of graded dimension $1 + q^2$, however, without a fixed basis. More precisely, we fix one element of the basis, denoted by 1, of grading 0. The other element of the basis is defined up to $\pm$ sign.

   Given an orientation o of the circle C , we associate with it the graded vector space generated by 1 and $X_{C,o}$. If we change the orientation to the opposite one $-o$, we set $X_{C,-o} = -X_{C,o}$.

2. Given a state s of the cube with k circles $C_1, \dots, C_k$ in it. With this state, we associate the ordered tensor product $V^{\otimes k}$ to it, the basis elements of this space being products $(p^1)_{C_{a_1}} \wedge (p^2)_{C_{a_2}} \wedge \dots \wedge (p^k)_{C_{a_k}}$, where each (p^i) is an element of the basis of $V_{C_{a_k}}$; here $p_{C_i} \wedge q_{C_j} = -q_{C_j} \wedge p_{C_i}$.

Now, the differentials are defined with respect to the orientations of the circles at vertices and local ordering of the components, as follows. For each vertex v, we fix the orientations of the circles incident to this vertex according to the orientation of the upper-right outgoing edge, see Fig 4.

That is, the orientation of these circles agrees with the orientation of upper-right and lower-right edges on the knot diagram and disagrees with that for the upper-left and lower-left edges: **We orient half-edges incident to a crossing as shown in the lower-left picture of Fig. 4; we say that the orientation of a state circle incident to it is positively oriented if it**

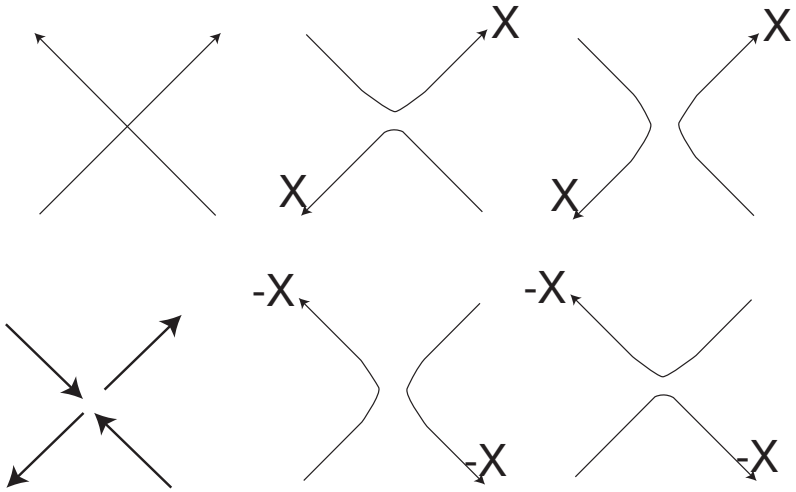

Figure 4: Determining the basis for a given vertex

agrees with the local orientations indicated in the lower-left picture; otherwise, the circle is negatively oriented.

This orientation is well-defined unless the edge corresponding to this vertex transforms one circle to one circle. **For such situations, we do not need to define X at this vertex; we just set the partial differential corresponding to this edge to be identically zero**.

Assume we have a $1 \to 2$ or $2 \to 1$-bifurcation at a vertex X.

If we deal with two vertices adjacent to a vertex from opposite sides, we enumerate them so that the upper [resp., left] one is denoted as the first one, and the lower [resp., right] one is the second one.

Agreement. Later on, while defining m and Δ we assume that the circles we deal with are in the very first position in our ordered tensor product. Otherwise, we permute them to get the desired expression, possibly, with minus sign.

Now, we define Δ and m locally with respect to the prescribed choice of generators in V's and the prescribed ordering.

$\Delta(1) = 1_1 \wedge X_2 + X_1 \wedge 1_2; \Delta(X) = X_1 \wedge X_2$ and

$m(1_1 \wedge 1_2) = 1; m(X_1 \wedge 1_2) = m(1_1 \wedge X_2) = X; m(X_1 \wedge X_2) = 0$, see Fig. 5.

Thus, if we wish to comultiply the second factor X_2 in $X_1 \wedge X_2$, we get $X_1 \wedge X_2 = -X_2 \wedge X_1 \to -X_2 \wedge X_3 \wedge X_1 = -X_1 \wedge X_2 \wedge X_3$, where X_3 belongs to the new (third) component.

3 Formulation and proof of the main theorem

Denote the obtained complex for a virtual diagram K, by $[[K]]$.

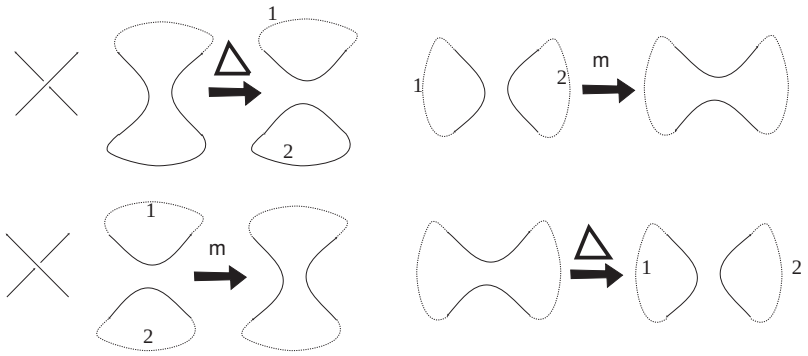

Figure 5: Definition of m and Δ

Theorem 2. *The complex $[[K]]$ described above, is well-defined for every virtual diagram K.*

We have to prove that the complex we obtain is well-defined.

Then, as in [Ma2], we normalize it by setting $\mathcal{C} := [[L]][-n_-]\{n_+ - 2n_-\}$. The obtained complex (that differs from the unnormalized one just by a grading and degree shift) will be well-defined as well. We will show that its homology is invariant under Reidemeister moves, this homology coincides with the one constructed in [Ma2] in the orientable case.

When considered modulo 2, the definition agrees with the one given in [Ma1] verbatim.

We first prove two lemmas establishing some properties of the complex and making the further arguments easier.

Given a virtual diagram K and a classical crossing V of it. Consider the diagram K' obtained from K by virtualizing V. There is a natural one-to-one correspondence between classical crossings of K and K'. For each classical crossing U of K, we denote the space corresponding to it by C_U; denote the corresponding space for K' by $C_{U'}$.

Lemma 1. *Let K, K' be two knot diagrams obtained one from another by a virtualization. Then there is a grading-preserving chain map $C(K) \to C(K')$ that maps C_{U_K} isomorphically to $C_{U_{K'}}$ and agrees with the local differentials.*

In particular, if $C(K)$ is a well-defined complex, then so is $C(K')$ and their homology groups are isomorphic.

Proof. Suppose a diagram K' is obtained from a diagram K by virtualization at a crossing U. Consider the corresponding cubes $C(K)$ and $C(K')$. Obviously, their differentials agree for all edges corresponding to all crossing except U. We assume the differentials corresponding to U divide our cubes into "top layer" and the "bottom layer", as shown in Fig. 6.

Now, the remaining differentials differ possibly by signs on edges corresponding to the crossing U. Our goal is to show that they either all agree or all differ

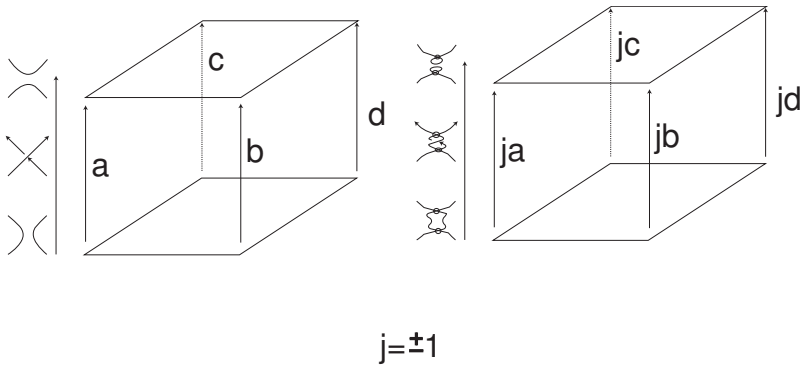

Figure 6: Behaviour of the cube under virtualization

by -1 sign, as shown in Fig. 6.

Indeed, the bases at all crossings but U agree for K and K'. This leads to the identification of chains of the corresponding complexes. Under this isomorphism, we see that for every circle incident to the crossing U, $X_{U,K} = -X_{U,K}$.

If we dealt with the usual tensor product case regardless the circle ordering, the transformation $X \to -X$ would leave m invariant and change Δ to $-\Delta$.

But we shall also take into account the circle ordering at a vertex.

First assume X is positive. Then all mappings m corresponding to X represent a multiplication of elements corresponding to two circles, a left one and a right one. After a virtualization, the former left circle becomes the right one, and the former right one goes to the left, see Fig. 7.

Globally, thus, it changes the sign of m in the fixed basis. For the case of Δ, we deal with one circle which is transformed to two circles, an upper one and a lower one; the relation "upper-lower" is preserved by virtualization. The first component above is represented by a solid line; the second component is dashed.

Summing up, we see that a virtualization at a positive crossing changes all signs of local differential corresponding to this vertex.

Now, divide the cubes $C(K)$ and $C(K')$ into two parts with respect to the smoothing of U, the lower one and the upper one. Define the mapping $C(K) \to C(K')$ to be identity for all elements from the lower cube and minus identity from all elements of the upper cube.

Evidently, this mapping agrees with local differentials, and, if the initial cube were commutative, this mapping would provide a homology isomorphism.

A similar argument for negative crossings show that in this case the cube is not changed at all: a minus sign which appears for Δ is then cancelled by the permutation of the two circles (left-right) in the target space.

This completes the proof of the lemma. □

This lemma means that the homology of a virtual diagram with two classical crossings (if well-defined) can be restored from an atom endowed with orienation

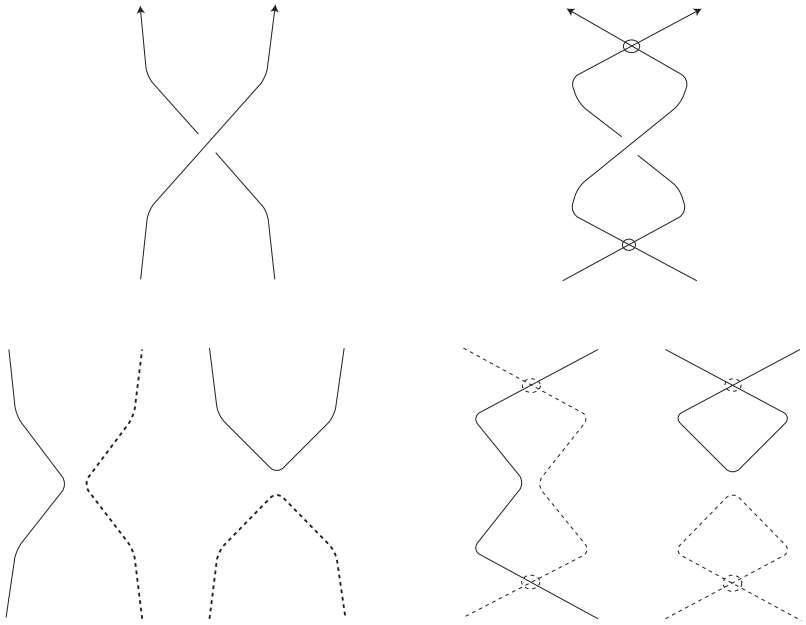

Figure 7: A virtualization

of the link components.

Thus, to prove that the cube of a diagram K' anticommutes, we can make some preliminary virtualizations for vertices of K.

To check the anticommutativity of the cube $C(K)$, we have to consider all 2-faces of it.

Each 2-face is represented by fixing a way of smoothing some $(n-2)$ vertices of K, see Fig.8. The remaining two crossings can be smoothed arbitrarily; the four possibilities correspond to the vertices of the state.

Now, for these four states, there are some "common" circles which do not touch any of the two vertices in question. After removing these circles, we get an atom with two vertices.

What we actually have to check is that any face corresponding to any possible atom with 2 vertices anti-commutes.

For two vertices of such an atom, we have some local orientations of the link at each of these vertices; they are needed to fix the local ordering of components while defining the differentials.

Note that globally these orientations might not agree on the circles; namely, an edge of the atom with 2 vertices consists of several edges of the diagram which might have opposite orientations, see Fig. 9.

It turns out, however, that these local orientations can be chosen arbitrarily without loosing the anticommutativity property and without changing the homology.

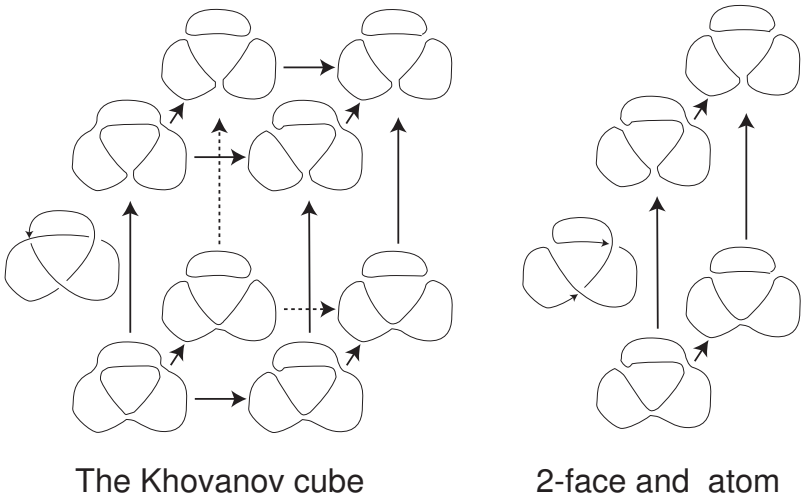

Figure 8: A 2-face generates an atom

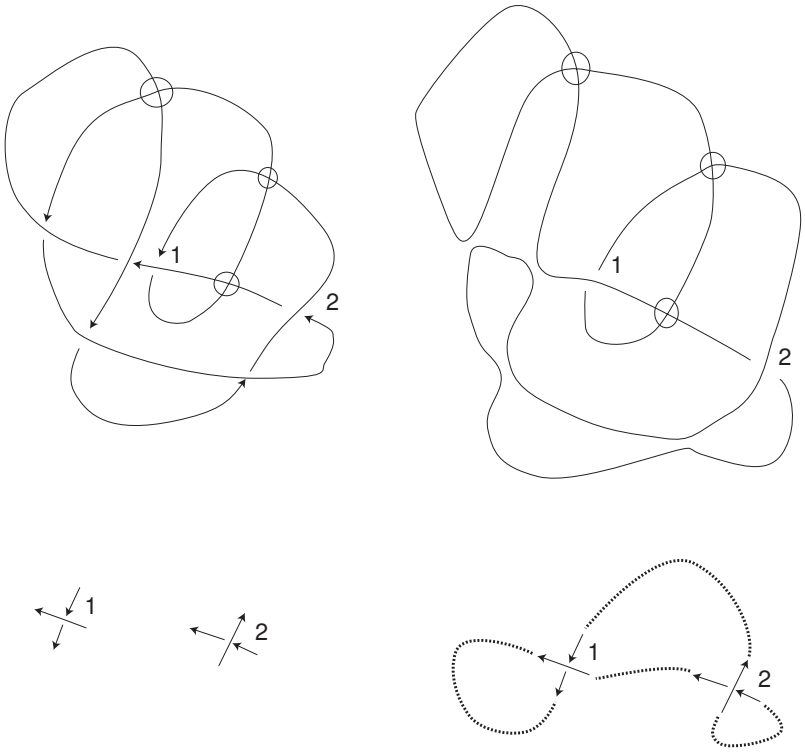

Figure 9: Orientation for atom crossings

Fix an atom with two vertices. All possible occurences of this atom in a cube correspond to local orientations of edges at these vertices. Fix an orientation for one vertex V_1 and choose two orientations for the second vertex V_2 that differ from each other by a $\frac{\pi}{2}$-turn. Thus, we get two pictures and two 2-dimensional discrete cubes, Q_1 and Q_2.

Lemma 2. *If Q_1 is anticommutative, then so is Q_2.*

Proof. Indeed, after a clockwise rotation at V_2, we change the sign of X at this crossing [again, we consider two complexes and identify their chains so that all differentials corresponding to the remaining vertices coincide]. This would lead to the overall sign change of Δ if we dealt with the the usual tensor powers.

Now, for a positive crossing, the sign of all m's corresponding to it changes as well.

For a negative crossing, the sign of all multiplications (m) is preserved whence the sign of Δ is changed for the second time. So, the situation is as in Lemma 1. □

This lemma means that in order to check the anticommutativity of all possible faces, it is sufficient to enumerate all atoms with 2 vertices and check the commutativity for each of them: we first fix an representation of such an atom in $\mathbf{R}^2$ [i.e., an immersion of its frame preserving the A-structure]; such immersions differ by a possible virutalization (which does change the homology by Lemma 1); then we choose a local orientation, which does not matter either by Lemma 2.

First, note that among atoms with two vertices, there are disconnected ones, i.e. those for which no edge connects one vertex to the other.

For such atoms, if we dealt with usual tensor powers of V, we would evidently get commutative faces. But the ordered tensor powers obviously make the situation anticommute.

Some (connected) atoms with two vertices are inessential in the following sense. We have defined the $1 \to 1$ differential to be zero. By parity reasons, in the 2-face of an atom there might be $0, 2$, or 4 such edges. The case when we have no such edges is orientable. When we have 4 all-to-zero mappings, there is nothing to prove. There is nothing to prove, either, when there is one all-to-zero mapping in one composition and one all-to-zero mapping in the other composition.

There are six essential connected atoms with two vertices, as shown in Fig. 10. All atoms except the first one are orientable.

For the first one, an accurate calculation corresponding Fig. 11 shows that both compositions give zero.

Indeed, starting with X, we get $\pm X \wedge X$ at the first step and 0 at the second step. If we start with 1, we get $1_{1,V_1} \wedge X_{2,V_2} + X_{1,V_1} \wedge 1_{2,V_2}$; here the first index is the number of the circle, and the second index is the name of the vertex. For the second vertex V_2, the first and the second circle change their roles: 1 becomes the lower one and 2 becomes the upper one. Also, for the second circle,

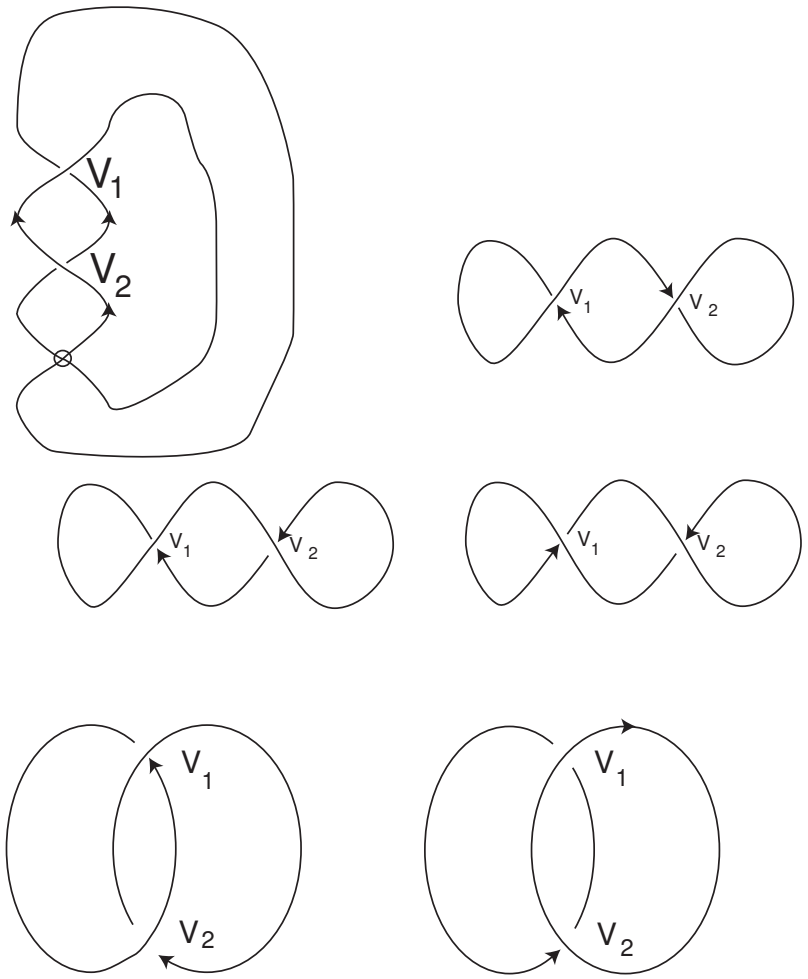

Figure 10: Essential atoms with 2 vertices

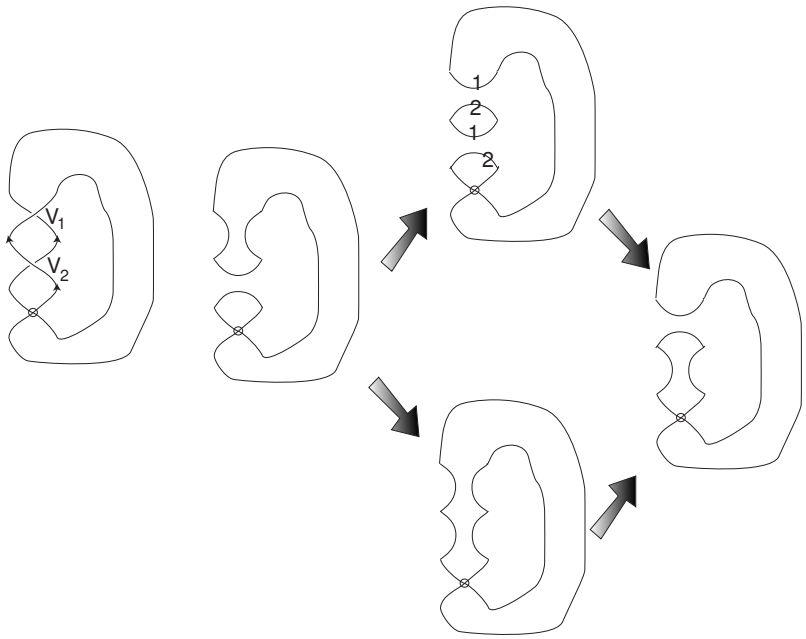

Figure 11: The non-orientable atom

X changes to $-X$. Thus we get $X \wedge 1 - 1 \wedge X$ which is transformed by m to zero.

For all orientable atoms, we fix the orientation as shown in Fig.10. Then, the the bases $\{1, X\}$ for vertices taken according to 4, will define an orientation of any circle.

Now, the anticommutativity is checked as follows. If we dealt with the tensor product case, everything would commute. Now, the enumeration of circles might cause minus signs on some edges. We have to check that for any of these five atoms the total sign would be minus.

For instance, in Fig. 12 we have an oriented atom with two vertices. The analogous check of the unordered tensor product case means the usual associativity $m \circ (m \otimes 1) = m \circ (1 \otimes m)$, where the circles are numbered from the left to the right.

In the left part of the figure, one pair 1 2 is drawn upside down to underline which circle is assumed to be locally the first (left); the other one is the second (right).

Here we have to take into account the global ordering of the components. Note that for three components, we have to apply always $m \wedge Id$ first, taking those components to be multiplied to the first and the second position.

Thus, $m \circ (m \wedge Id)$ applied to $A_1 \wedge A_2 \wedge A_3$ gives us $m(m(A_1, A_2), A_3) = -(A_1 \cdot A_2 \cdot A_3)$; here $\cdot$ means the usual multiplication in Khovanov's sense: $X \cdot X = 0; X \cdot 1 = 1 \cdot X = X; 1 \cdot 1 = 1$. Here the minus sign appears because

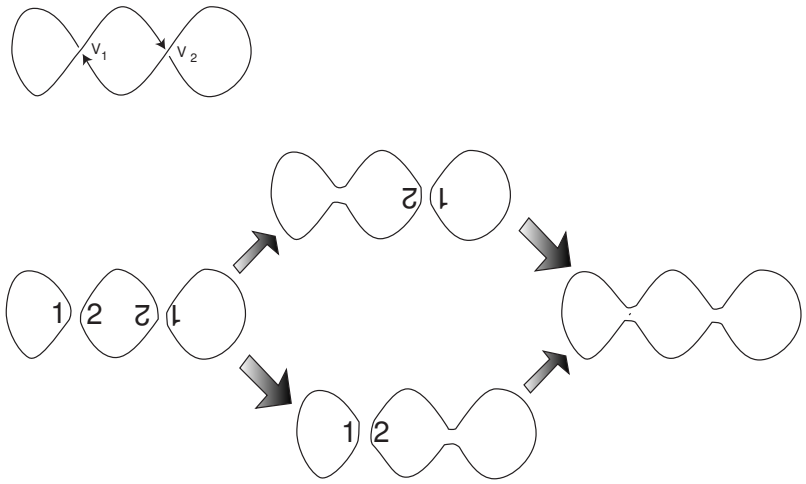

Figure 12: An orientable 2-vertex atom

at the second crossing, we have two branches oriented downwards, thus, the rightmost circle occurs to be locally the left one.

On the other hand, if we consider the second crossing V_2 first, we get $A_1 \wedge A_2 \wedge A_3 = (A_2 \wedge A_3) \wedge A_1 = -(A_3 \wedge A_2) \wedge A_1 \to -(A_2 \cdot A_3) \wedge A_1 = A_1 \wedge (A_2 \cdot A_3)$. Applying m to that, we get $A_1 \cdot A_2 \cdot A_3$.

All other atoms are checked analogously. Note that our setup gives directly an anticommutative cube, unlike the Khovanov original setup, where we got an anticommutative cube from a commutative one by adding some minus signs on vertices.

Theorem 3. *$Kh(K)$ is invariant under Reidemeister moves.*

In fact, this theorem goes in the same lines as in [BN]; one should just take care about signs of some local differentials.

Theorem 4. *Let K be a virtual diagram with orientable corresponding atom. Then the homology $Kh(K)$ coincides with the usual Khovanov homology [Ma2].*

During the proof of this theorem, we denote our complex and our homology by $C(K)$ and $Kh(K)$, and the ones constructed in [Ma2] by $C'(K)$ and $Kh'(K)$, respectively.

Proof. First, we assume the diagram of K is chosen in such a way that all X's for all vertices and circles agree. This is possible since the atom corresponding to K is orientable.

After that, we should just care about signs of local differential and enumeration of circles for any vertex.

The plan is to construct a homology-preserving mapping between the two cubes. First, $C'(K)$ does depend on enumeration of crossings. Let us fix

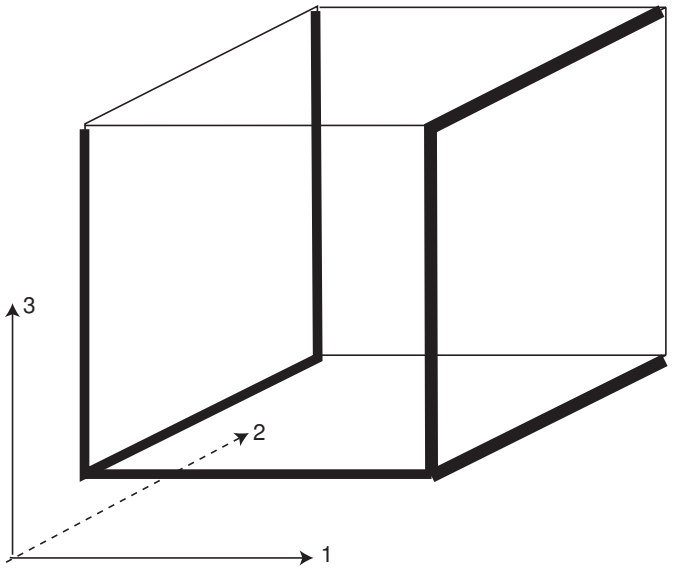

Figure 13: Choosing a spanning tree

such an enumeration. With this enumeration, associate a maximal spanning tree for the cubes $C(K)$ and $C'(K)$. This tree consists of all edges of type $(\alpha_1, \ldots, \alpha_k, *, 0, \ldots, 0), \alpha_j \in \{0, 1\}$, that is, an edge in direction x_j is chosen if and only if all other coordinates $x_j + 1, \ldots, x_n$ are zero, see Fig.13.

Now, for any state s of the cube $C(K)$, we have the signed tensor power $V^{\wedge k}$ and $V^{\otimes k}$ for $C'(K)$, where k denotes the number of circles. Enumerate the circles in the A-state (the lowest corner of the cube) somehow. This ordering defines a mapping from the space corresponding to the A-state s in $C(K)$ to the space corresponding to $C'(K)$. We may prolongate this mapping to all states of the cube so that it agrees with the differentials local along the edges of the spanning tree.

Now, this mapping indeed agrees with all the remaining edges because of the anticommutativity of both cubes.

□

4 Post Scriptum

The construction presented above gives a partial solution to the question: Is it true that any two classical link diagrams K and K', which can be connected by a sequence of generalized Reidemeister moves and virtualizations, are classically equivalent. Indeed, such diagrams should have the same Khovanov homology.

Several constructions and results concerning the Khovanov homology generalize straightforwardly for the theory presented here.

For instance, so is the spanning tree expansion, see [Weh] and all minimality results, see [Ma3].

More precisely, the thickness of the Khovanov homology of a virtual knot does not exceed $2 + g$, where g is the genus of any atom representing K.

Also, Khovanov's Frobenius theory with basic ring $\mathcal{R}$ and the homology of the unknot $\mathcal{A}$,

1. $\mathcal{R} = \mathbf{Z}[\langle, \sqcup]$.
2. $\mathcal{A} = \mathcal{R}[\mathcal{X}]/(\mathcal{X}^{\in} - \langle\mathcal{X} - \sqcup)$,
3. $\deg X = 2, \deg h = 2, \deg t = 4$;
4. $\Delta(1) = 1 \otimes X + X \otimes 1 - h1 \otimes 1$
5. $\Delta(X) = X \otimes X + t1 \otimes 1$,

admits a straightforward generalization as above for $h = 1$ [with $1 \to 1$-differential being zero. In particular, it leads to a generalization of Lee's theory [Lee].

We shall discuss other aspects of this theory, in particular, the general case of Frobenius' extensions and the relation of this theory to Bar-Natan's theory for tangles and cobordisms, see [BN2, TuTu], and other connections to Khovanov-Rozansky homology theory.

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